THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I
Suggested Solutions for HW1

Field Axioms of real number:

A1. $a + b \in \mathbb{R}$ if $a, b \in \mathbb{R}$;

A2. $a + b = b + a$ if $a, b \in \mathbb{R}$;

A3. $a + (b + c) = (a + b) + c \in \mathbb{R}$ if $a, b, c \in \mathbb{R}$;

A4. There exists $0 \in \mathbb{R}$ such that $a + 0 = a$ for all $a \in \mathbb{R}$;

A5. For any $a \in \mathbb{R}$, there is $b \in \mathbb{R}$ such that $a + b = 0$;

M1. $a \cdot b \in \mathbb{R}$ if $a, b \in \mathbb{R}$;

M2. $a \cdot b = b \cdot a$ if $a, b \in \mathbb{R}$;

M3. $a \cdot (b \cdot c) = (a \cdot b) \cdot c \in \mathbb{R}$ if $a, b, c \in \mathbb{R}$;

M4. There exists $1 \in \mathbb{R} \setminus \{0\}$ such that $a \cdot 1 = a$ for all $a \in \mathbb{R}$;

M5. For any $a \in \mathbb{R} \setminus \{0\}$, there is $b \in \mathbb{R}$ such that $a \cdot b = 1$;

D. $a \cdot (b + c) = a \cdot b + a \cdot c$ if $a, b, c \in \mathbb{R}$.

Order axioms of real number:

There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of positive real numbers, such that:

O1. If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$.

O2. If $a, b \in \mathbb{P}$, then $a \cdot b \in \mathbb{P}$.

O3. (Trichotomy property) If $a \in \mathbb{R}$, then exactly one of the following holds:

$$a \in \mathbb{P}, \quad a = 0, \quad -a \in \mathbb{P}.$$

1. Using the Axioms, show that
   (a) for all $a \in \mathbb{R} \setminus \{0\}$, $1/(1/a) = a$,
   (b) If $a > b > 0$, then $0 < a^{-1} < b^{-1}$.

Solution. (a) We first show the uniqueness of multiplicative inverses given in (M5). Let $a \in \mathbb{R} \setminus \{0\}$. Suppose both $b, c \in \mathbb{R}$ such that $a \cdot b = 1$ and $a \cdot c = 1$. We want to show that $b = c$.

$$\begin{aligned}
b &= b \cdot 1 && \text{(M4)} \\
&= b \cdot (a \cdot c) && \text{(assumption)} \\
&= (b \cdot a) \cdot c && \text{(M3)} \\
&= (a \cdot b) \cdot c && \text{(M2)} \\
&= 1 \cdot c && \text{(assumption)} \\
&= c \cdot 1 && \text{(M2)} \\
&= c && \text{(M4)}.
\end{aligned}$$

Since multiplicative inverses are unique, we call it $\frac{1}{a}$. We now show that $1/(1/a) = a$. Replacing a with $1/a$, we know that $1/(1/a)$ is the multiplicative inverse of $1/a$, so we have

$$\begin{aligned}
\frac{1}{1/a} &= \frac{1}{1/a} \cdot 1 && \text{(M4)} \\
&= \frac{1}{1/a} \cdot \left(a \cdot \frac{1}{a}\right) && \text{(M5)} \\
&= \frac{1}{1/a} \cdot \left(\frac{1}{a} \cdot a\right) && \text{(M2)} \\
&= \left(\frac{1}{1/a} \cdot \frac{1}{a}\right) \cdot a && \text{(M3)} \\
&= \left(\frac{1}{a} \cdot \frac{1}{1/a}\right) \cdot a && \text{(M2)} \\
&= 1 \cdot a && \text{(M5)} \\
&= a \cdot 1 && \text{(M2)} \\
&= a && \text{(M4)}
\end{aligned}$$

as required.

(b) We first show the following:

i. Uniqueness of additive inverse: Let $a \in \mathbb{R}$ and suppose both $b, c \in \mathbb{R}$ such that $a + b = 0$ and $a + c = 0$. We want to show that $b = c$.

$$\begin{aligned}
b &= b + 0 && \text{(A4)} \\
&= b + (a + c) && \text{(assumption)} \\
&= (b + a) + c && \text{(A3)} \\
&= (a + b) + c && \text{(A2)} \\
&= 0 + c && \text{(assumption)} \\
&= c + 0 && \text{(A2)} \\
&= c.
\end{aligned}$$

Since additive inverses are unique, we call it $-a$.

ii. $0 = a \cdot 0$ for all $a \in \mathbb{R}$:

$$\begin{aligned}
0 &= a \cdot 0 + (-a \cdot 0) && (A5, \text{(i) above}) \\
&= a \cdot (0 + 0) + (-a \cdot 0) && \text{(A4)} \\
&= a \cdot 0 + a \cdot 0 + (-a \cdot 0) && \text{(D)} \\
&= a \cdot 0 && \text{(A5).}
\end{aligned}$$

iii. $a \cdot (-1) = -a$ for all $a \in \mathbb{R}$:

$$\begin{aligned}
0 &= a \cdot 0 && \text{((ii) above)} \\
&= a \cdot (1 + (-1)) && \text{(A5, (i) above)} \\
&= a \cdot 1 + a \cdot (-1) && \text{(D)} \\
&= a + a \cdot (-1) && \text{(M4).}
\end{aligned}$$

So $a \cdot (-1)$ is such that $a + a \cdot (-1) = 0$, so by (i) above, $a \cdot (-1) = -a$.

iv. For any $a \in \mathbb{R}$, define $a^2 := a \cdot a$. Then show that $(-1)^2 = 1$:

$$\begin{aligned}
(-1)^2 &= (-1)^2 + 0 && \text{(A4)}\\
&= (-1)^2 + (-1) + 1 && \text{(A5)}\\
&= (-1)\cdot(-1) + (-1) + 1 && \text{(definition of square)}\\
&= (-1)\cdot(-1) + (-1)\cdot 1 + 1 && \text{(M4)}\\
&= (-1)\cdot((-1)+1) + 1 && \text{(D)}\\
&= (-1)\cdot(1+(-1)) + 1 && \text{(A2)}\\
&= (-1)\cdot 0 + 1 && \text{(A5)}\\
&= 0 + 1 && \text{((ii) above)}\\
&= 1 && \text{(A2,A4).}
\end{aligned}$$

v. for all $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$: Since $a \neq 0$, by the Trichotomy property, either $a \in \mathbb{P}$ or $-a \in \mathbb{P}$. Then for the case where $a \in \mathbb{P}$, we have $a^2 = a \cdot a \in \mathbb{P}$ by O1. For the case where $-a \in \mathbb{P}$, we have

$$\begin{aligned}
(-a)^2 &= (-a)\cdot(-a) && \text{(definition of square)}\\
&= (a\cdot(-1))\cdot(a\cdot(-1)) && \text{((iii) above)}\\
&= (a\cdot(-1))\cdot((-1)\cdot a) && \text{(M2)}\\
&= a\cdot((-1)\cdot(-1))\cdot a && \text{(M3)}\\
&= a\cdot(-1)^2\cdot a && \text{(definition of square)}\\
&= a\cdot 1\cdot a && \text{((iv) above)}\\
&= a\cdot a && \text{(M4)}\\
&= a^2
\end{aligned}$$

and since $(-a) \in \mathbb{P}$, we see that $a^2 = (-a)\cdot(-a) \in \mathbb{P}$. Therefore, we have that $a^2 > 0$.

vi. $1 > 0$: By (M4), we know that $1 \neq 0$. We have

$$\begin{aligned}
1 &= 1\cdot 1 && \text{(M4)}\\
&= 1^2 && \text{(definition of square)}\\
&> 0 && \text{((v) above).}
\end{aligned}$$

vii. If $a > b$ and $c > 0$, then $c\cdot a > c\cdot b$, and if $a > b$ and $c < 0$, then $c\cdot a < c\cdot b$: We know that $a > b$ means $a + (-b) \in \mathbb{P}$, and $c > 0$ means $c \in \mathbb{P}$. So we have

$$\begin{aligned}
c\cdot a + (-c\cdot b) &= c\cdot a + ((-1)\cdot c\cdot b) && \text{((iii) above}\\
&= c\cdot a + (c\cdot b\cdot(-1)) && \text{(M2 twice)}\\
&= c\cdot a + (c\cdot(-b)) && \text{((iii) above)}\\
&= c\cdot(a+(-b)) && \text{(D)}\\
&> 0 && \text{(O2).}
\end{aligned}$$

Now consider the case where $a > b$ and $c < 0$, that is, that $-c \in \mathbb{P}$. Then we have

$$\begin{aligned}
c \cdot b + (-c \cdot a) &= (-c \cdot a) + c \cdot b && \text{(A2)} \\
&= (-c \cdot a) + 1 \cdot c \cdot b && \text{(M2,M4)} \\
&= (-c \cdot a) + (-1)^2 \cdot c \cdot b && \text{((iv) above)} \\
&= (-c \cdot a) + (-1) \cdot (-1) \cdot c \cdot b && \text{(definition of square)} \\
&= (-c \cdot a) + (-1) \cdot (-c) \cdot b && \text{((iii) above)} \\
&= (-c \cdot a) + ((-c) \cdot (-1) \cdot b) && \text{(M2)} \\
&= (-c) \cdot a + ((-c) \cdot b \cdot (-1)) && \text{(M2,M3 above)} \\
&= (-c) \cdot a + ((-c) \cdot (-b)) && \text{((iii) above} \\
&= (-c) \cdot (a + (-b)) && \text{(D)} \\
&> 0 && \text{(O2).}
\end{aligned}$$

viii. If $a > 0$, then $1/a > 0$: If $a > 0$, then by the Trichotomy property, $a \neq 0$, therefore, $1/a$ exists. Suppose that $1/a = 0$, then

$$\begin{aligned}
1 &= a \cdot \frac{1}{a} && \text{(M5)} \\
&= a \cdot 0 && \text{(by assumption)} \\
&= 0 && \text{((ii) above)}
\end{aligned}$$

a contradiction. On the other hand, suppose that $1/a < 0$, then

$$\begin{aligned}
1 &= a \cdot \frac{1}{a} && \text{(M5)} \\
&< 0 && \text{((vii) above)}
\end{aligned}$$

which contradicts (vi) above.

Finally we are able to show the main result, which is that if $a > b > 0$, then $0 < a^{-1} < b^{-1}$, where we understand a^{-1} to be another notation for $1/a$. By (viii) above, we see that both $a^{-1}, b^{-1} > 0$. We have:

$$\begin{aligned}
0 < b < a &\implies 0 \cdot b^{-1} < b \cdot b^{-1} < a \cdot b^{-1} && \text{((vii) above)} \\
&\implies 0 < 1 < a \cdot b^{-1} && \text{((ii) above, M5)} \\
&\implies a^{-1} \cdot 0 < a^{-1} \cdot 1 < a^{-1} \cdot a \cdot b^{-1} && \text{(((vii) above)} \\
&\implies 0 < a^{-1} b^{-1} && \text{(M4, M5, M2)}
\end{aligned}$$

as required.

◀

2. If A is a non-empty subset of $\mathbb{R}$ such that A is bounded from above. If we denote $-A = \{-a : a \in A\}$, show that $\inf(-A)$ exists and equals to $-\sup A$.

Solution. Since A is non-empty and bounded from above, the set $-A$ is non-empty and bounded from below. Hence, by the completeness of $\mathbb{R}$, $\inf(-A)$ exists.

It remains to show that $\inf(-A) = -\sup A$. Let $u = \sup A$. We want to show that $-u = \inf(-A)$.

Lower bound: Since $u = \sup A$, we know that $a \leq u$ for all $a \in A$. Multiplying by -1, we see that $-u \leq -a$ for each $a \in A$ and hence $-u$ is a lower bound of $-A$.

Greatest lower bound property: Let v be a lower bound of $-A$. Then for any $b \in -A$, we know that $v \leq b$. Note that $-b \in A$, so multiplying by -1 we see that $-b \leq -v$. Since u is the supremum of A, we have that $-b \leq u \leq -v$. Multiplying again by -1 we have $v \leq -u \leq b$ as required. ◀

3. Show that if A, B are bounded subset of $\mathbb{R}$. Show that

$$\sup(A + B) = \sup A + \sup B$$

where $A + B = \{a + b : a \in A, b \in B\}$. Do we have

$$\sup A \cdot \sup B = \sup(A \cdot B)$$

where $A \cdot B = \{ab : a \in A, b \in B\}$? Justify your answer.

Solution. We will show that $\sup(A + B) \leq \sup A + \sup B$ and $\sup A + \sup B \leq \sup(A + B)$.

$\sup(A + B) \leq \sup A + \sup B$: let $a \in A$ and $b \in B$. We know that $a \leq \sup A$ and $b \leq \sup B$, so adding these two inequalities together we have $a + b \leq \sup A + \sup B$. Since a and b were arbitrary, the element $a + b$ was arbitrarily chosen and so the number $\sup A + \sup B$ is an upper bound of $A + B$. Hence $\sup(A + B) \leq \sup A + \sup B$.

$\sup A + \sup B \leq \sup(A + B)$: let $a \in A$. Then for all $b \in B$, $a + b \in A + B$ and we know that $a + b \leq \sup(A + B) \implies b \leq \sup(A + B) - a$. Since this inequality holds for all $b \in B$, this means the number $\sup(A + B) - a$ is an upper bound of the set B, hence we have $\sup B \leq \sup(A + B) - a$. Rearranging gives us $a \leq \sup(A + B) - \sup B$. Since a was chosen arbitrarily, this means the number $\sup(A + B) - \sup B$ is an upper bound for the set A and we have $\sup(A) \leq \sup(A + B) - \sup B$. Rearranging the inequality gives the result.

No, we do not have $\sup A \cdot \sup B = \sup(A \cdot B)$. Consider $A = \{-1, 1\}, B = \{-2, 1\}$. Then $\sup A = 1, \sup B = 1$, but $\sup(A \cdot B) = 2$. ◀

4. Let X be a non-empty set and $f, g : X \to \mathbb{R}$ be two real valued function with bounded ranges. Show that

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

Give an example showing that the inequality can be a strict inequality.

Solution. Since f, g have bounded ranges in $\mathbb{R}$, the supremums exist. Let $u = \sup\{f(x) : x \in X\}$ and $v = \sup\{g(x) : x \in X\}$. Then for all $x \in X$, $f(x) \leq u$ and $g(x) \leq v$. Adding these two inequalities together, we have

$$f(x) + g(x) \leq u + v$$

and hence $u + v$ is an upper bound of the set $\{f(x) + g(x) : x \in X\}$. Then by definition of supremum, we have

$$\sup\{f(x) + g(x) : x \in X\} \leq u + v = \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

as required.

For the example of strict inequality, consider $X = [-1, 1]$ and set $f(x) = x$, $g(x) = -x$. Then $f(x) + g(x) = 0$, so $\sup\{f(x) + g(x) : x \in X\} = 0$, while $\sup\{f(x) : x \in X\} = 1$ and $\sup\{g(x) : x \in X\} = 1$ and so $\sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\} = 2$. ◀

5. Show by using completeness that there is $x \in \mathbb{R}$ so that $x > 0$ and $x^3 + x = 5$. Show that such x is unique.

Solution. Let $S := \{s \in \mathbb{R} : s^3 + s < 5\}$. Since $1 \in S$, S is not empty. Moreover, S is bounded from above by 5. So by the completeness of $\mathbb{R}$, $\sup S$ exists in $\mathbb{R}$ and moreover, $x := \sup S \geq 1 > 0$.

Suppose $x^3 + x < 5$. Then by assumption, $5 - x^3 - x > 0$ and since $x > 0$, we also have $3x^2 + 3x + 2 > 0$. Then by the Archimedean property, we can find an $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \frac{5 - x^3 - x}{3x^2 + 3x + 2}.$$

Then since $\frac{1}{n^3} \leq \frac{1}{n}, \frac{1}{n^2} \leq \frac{1}{n}$ and since $x > 0$, we have

$$\begin{aligned}
\left(x + \frac{1}{n}\right)^3 + \left(x + \frac{1}{n}\right) &= x^3 + \frac{3x^2}{n} + \frac{3x}{n^2} + \frac{1}{n^3} + x + \frac{1}{n} \\
&\leq x^3 + \frac{3x^2}{n} + \frac{3x}{n} + \frac{1}{n} + x + \frac{1}{n} \\
&= x^3 + x + \frac{1}{n}\left(3x^2 + 3x + 2\right) \\
&< x^3 + x + \left(\frac{5 - x^3 - x}{3x^2 + 3x + 2}\right)\left(3x^2 + 3x + 2\right) = 5.
\end{aligned}$$

So $\left(x + \frac{1}{n}\right) \in S$, which contradicts the fact that x is an upper bound of S. Hence $x^3 + x < 5$ is not possible.

Suppose on the other hand that $x^3 + x > 5$. Then by assumption, $x^3 + x - 5 > 0$ and since $x > 0$, we also have $3x^2 + 2 > 0$. Then by the Archimedean property, we can find an $m \in \mathbb{N}$ such that

$$\frac{1}{m} < \frac{x^3 + x - 5}{3x^2 + 2}.$$

Then since $\frac{1}{m^3} \leq \frac{1}{m}$ and since $x > 0$, we have

$$\begin{aligned}
\left(x - \frac{1}{m}\right)^3 + \left(x - \frac{1}{m}\right) &= x^3 - \frac{3x^2}{m} + \frac{3x}{m^2} - \frac{1}{m^3} + x - \frac{1}{m} \\
&> x^3 + x - \frac{3x^2}{m} - \frac{1}{m} - \frac{1}{m^3} \\
&\geq x^3 + x - \frac{3x^2}{m} - \frac{2}{m} \\
&= x^3 + x - \frac{1}{m}\left(3x^2 + 2\right) \\
&> x^3 + x - \left(\frac{x^3 + x - 5}{3x^2 + 2}\right)\left(3x^2 + 2\right) = 5.
\end{aligned}$$

So $\left(x - \frac{1}{m}\right)$ is an upper bound of S, which contradicts the fact that x is the least upper bound of S. Hence $x^3 + x > 5$ is not possible.

So we have that $x^3 + x = 5$.

For uniqueness, suppose there is a $y \neq x$ such that $y^3 + y = 5$. Then we have

$$\begin{aligned}
0 &= 5 - 5 \\
&= x^3 + x - y^3 - y \\
&= x^3 - y^3 + x - y \\
&= (x - y)(x^2 + xy - y^2) + (x - y) \\
&= (x - y)(x^2 + xy - y^2 + 1).
\end{aligned}$$

So either $x - y = 0$ or $x^2 + xy - y^2 + 1 = 0$. If $x - y = 0$, then we would have $x = y$, a contradiction, and we are done. On the other hand, suppose $x^2 + xy - y^2 + 1 = 0$. The left hand side is a polynomial in x with determinant

$$\Delta = 5y^2 + 4 > 0, y \in \mathbb{R}$$

and so $x^2 + xy - y^2 + 1 = 0$ admits no real solutions. ◀